Effective Liouville Equation for Classical Driven Systems*

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A large class of classical dynamical systems with an external rapidly oscillating driving action is considered and the effective Hamiltonian-like equations for the mean motion are obtained. The respective Liouville equation for the distribution function of the mean coordinates and momenta is derived. PACS numbers 03.20.+i, 05.20.-y, 05.20.Gg

The inverted pendulum, studied by P.L. Kapitza [1] and recently revisited by S.-Y. Kim and B. Hu [2], is a very good example of systems where a fast driving external action imposes a complex dynamics to the system. The main result in Kapitza's theory is that a simple nonlinear oscillator under the action of a rapidly oscillating force is not conserving the mean motion on trajectories of a non perturbed analog. Instead of this, it behaves in a completely different manner, because an additional non small term proportional to the squared amplitude of external pulses appears in the restoring force. This additional restoring force is responsible for the stabilization of the inverted pendulum, which appears as a "miracle " and in no way could be predicted intuitively. It is therefore very tempting to look for such effects in extended many-body systems.

In this Letter, we propose an approach to the treatment of a large class of such systems in a Hamiltonian-like formalism, which provides a straightforward transition to the statistical thermodynamics of these systems.

Consider the following dynamical equations

$$\dot{\mathbf{q}}_i = \frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial \mathbf{p}_i} + \frac{\mathbf{g}_i^{(0)}(\mathbf{q},\mathbf{p})}{2} + \sum_{k\geq 1} \mathbf{g}_i^{(k)}(\mathbf{q},\mathbf{p})\cos(k\Omega t); \tag{1}$$

$$\dot{\mathbf{p}}_i = -\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial \mathbf{q}_i} + \frac{\mathbf{h}_i^{(0)}(\mathbf{q},\mathbf{p})}{2} + \sum_{k\geq 1} \mathbf{h}_i^{(k)}(\mathbf{q},\mathbf{p})\cos(k\Omega t), \quad i = 1,2,\ldots,N, \tag{2}$$

where $\mathbf{g}$ and $\mathbf{h}$ are arbitrary functions of all canonical variables. Here $\Omega \gg \omega_o$, where $2\pi/\omega_o$ is the characteristic time of the undriven system (with the Hamiltonian $H(\mathbf{q},\mathbf{p},t)$). The series in the right-hand side of (1), (2) can be considered as Fourier expansions of arbitrary vector functions of t with characteristic time $2\pi/\Omega$.

We do not require the system being necessarily Hamiltonian, i.e. the equations (1), (2) are of the general type $\dot{\mathbf{\Gamma}} = \mathcal{F}(\mathbf{\Gamma})$, where $\mathbf{\Gamma}(t) = (\mathbf{q},\mathbf{p})$ and $\mathcal{F}$ is a vector function of the phase space. The non-Hamiltonian nature of these equations means that the compressibility does not in general vanish:

$$\nabla_{\mathbf{\Gamma}} \cdot \dot{\mathbf{\Gamma}} = \nabla_{\mathbf{\Gamma}} \cdot \mathcal{F} \equiv \varkappa(\mathbf{\Gamma}) \neq \mathbf{0} \tag{3}$$

The non-Hamiltonian dynamics is useful when considering open or driven systems, under the action of some external influence, such as heat bath or mechanical piston, etc. M. E. Tuckerman, C.J. Mundy and M.L. Klein [3], [4] have recently derived a correct generalization of the Liouville equation to account for nonvanishing compressibility of phase space (3):

$$\frac{\partial\left(f\overline{J}\right)}{\partial t} = -\nabla_{\mathbf{\Gamma}} \cdot \left(\dot{\mathbf{\Gamma}}\, f\overline{J}\right), \tag{4}$$

where the Jacobian $\overline{J}$ satisfies the differential equation

$$\frac{d\overline{J}}{dt} = -\overline{J}\,\nabla_{\mathbf{\Gamma}} \cdot \dot{\mathbf{\Gamma}}\,. \tag{5}$$

Note that substituting (5) into (4) and taking into account the obvious relation $\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \dot{\mathbf{\Gamma}} \cdot \nabla_{\mathbf{\Gamma}} J$ leads to the conservation condition $df/dt = 0$ or

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$$\frac{\partial f}{\partial t}+\dot{\mathbf{\Gamma}}\cdot\nabla_{\mathbf{\Gamma}}f=0. \tag{6}$$

The backbone of our approach is the search of solutions of eqs. (1), (2) in the form

$$\mathbf{q}_i=\mathbf{Q}_i+\mu\chi_i; \tag{7}$$
$$\mathbf{p}_i=\mathbf{P}_i+\mu\rho_i, \tag{8}$$

where $\mathbf{Q}$, $\mathbf{P}$ and χ, ρ are respectively the "slow " and "fast " parts, whose characteristic times are accordingly $T\sim 2\pi/\omega_o$ and $\tau\sim 2\pi/\Omega$, and $\mu=\omega_o/\Omega\ll 1$. The expression (7) was successfully used [5] for the solution of the equation of motion of inverted pendulum. Such an ansatz (7)-(8) is physically justified, because due to inertia the system responds weakly to fast external pulses.

Substituting (7), (8) into (1), (2) and expanding all functions in power series of μ and retaining terms up to the first order (e.g. $\mathbf{g}_i^{(k)}(\mathbf{q},\mathbf{p})\simeq\mathbf{g}_i^{(k)}(\mathbf{Q},\mathbf{P})+\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\mathbf{g}_i^{(k)}(\mathbf{Q},\mathbf{P})+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\mathbf{g}_i^{(k)}(\mathbf{Q},\mathbf{P})$, etc.), we obtain

$$\begin{aligned}\dot{\mathbf{Q}}_i+\mu\,\dot{\chi}_i\ &=\frac{\partial H}{\partial\mathbf{P}_i}+\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\partial H}{\partial\mathbf{P}_i}+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\partial H}{\partial\mathbf{P}_i}+\frac{\mathbf{g}_i^{(0)}(\mathbf{Q},\mathbf{P})}{2}+\\&\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\mathbf{g}_i^{(0)}(\mathbf{Q},\mathbf{P})}{2}+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\mathbf{g}_i^{(0)}(\mathbf{Q},\mathbf{P})}{2}+\sum_{k\geq1}\mathbf{g}_i^{(k)}(\mathbf{Q},\mathbf{P})\cos(k\Omega t)+\\&\mu\sum_{k\geq1}(\chi_j\cdot\nabla_{\mathbf{Q}_j})\mathbf{g}_i^{(k)}(\mathbf{Q},\mathbf{P})\cos(k\Omega t)+\mu\sum_{k\geq1}(\rho_j\nabla_{\mathbf{P}_j})\mathbf{g}_i^{(k)}(\mathbf{Q},\mathbf{P})\cos(k\Omega t);\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{\mathbf{P}}_i+\mu\,\dot{\rho}_i\ &=-\frac{\partial H}{\partial\mathbf{Q}_i}-\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\partial H}{\partial\mathbf{Q}_i}-\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\partial H}{\partial\mathbf{Q}_i}+\frac{\mathbf{h}_i^{(0)}(\mathbf{Q},\mathbf{P})}{2}+\\&\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\mathbf{h}_i^{(0)}(\mathbf{Q},\mathbf{P})}{2}+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\mathbf{h}_i^{(0)}(\mathbf{Q},\mathbf{P})}{2}+\sum_{k\geq1}\mathbf{h}_i^{(k)}(\mathbf{Q},\mathbf{P})\cos(k\Omega t)+\\&\mu\sum_{k\geq1}(\chi_j\cdot\nabla_{\mathbf{Q}_j})\mathbf{h}_i^{(k)}(\mathbf{Q},\mathbf{P})\cos(k\Omega t)+\mu\sum_{k\geq1}(\rho_j\nabla_{\mathbf{P}_j})\mathbf{h}_i^{(k)}(\mathbf{Q},\mathbf{P})\cos(k\Omega t).\end{aligned} \tag{10}$$

(Here we use the Einstein summation convention on repeated indices). All functions in these equations depend on mean variables $(\mathbf{Q},\mathbf{P})$. The equations contain slow and fast terms which can be separated by averaging over the period $\tau=2\pi/\Omega$. More precisely, (9) and (10) are of the form: $Slow+Fast=0$. Since $\Omega\gg\omega_0$, the slow terms may be considered constant over the time $2\pi/\Omega$ and therefore may be substituted by their averages. The equations take the form: $\langle Slow\rangle_\Omega+Fast\simeq 0$. On the other hand, it is evident that $\langle Fast\rangle_\Omega=0$ and consequently the result is a set of coupled equations for the slow and fast variables: $\langle Slow\rangle_\Omega=0$; $Fast=0$. Hence, the first step consists in integrating (9), (10) with respect to time over the period $2\pi/\Omega$. Notice that $\mathbf{Q},\mathbf{P}$, as well as $H(\mathbf{Q},\mathbf{P},t)$ are considered to remain constant when integrating over "fast time ". Only slow terms withstand this operation. The remaining terms form the fast equations. Particular attention is to be given to the last and next to the last terms of (9), (10). These terms contain the products $\chi_j\cos(k\Omega t)$ and $\rho_j\cos(k\Omega t)$. Inasmuch as χ_j and ρ_j may contain both $\cos(k'\Omega t)$ and $\sin(k'\Omega t)$ (as will become apparent later), we have here the products $\cos(k'\Omega t)\cos(k\Omega t)$ and $\sin(k'\Omega t)\cos(k\Omega t)$. The average of the first product is nonzero for $k'=k$, whereas the average of the second product always vanishes. In other words, the last two terms in (9), (10) may in general contribute both to slow and fast equations. These latter thus read

$$\begin{aligned}\mu\,\dot{\chi}_i\ &=\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\partial H}{\partial\mathbf{P}_i}+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\partial H}{\partial\mathbf{P}_i}+\\&\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\mathbf{g}_i^{(0)}}{2}+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\mathbf{g}_i^{(0)}}{2}+\sum_{k\geq1}\mathbf{g}_i^{(k)}\cos(k\Omega t)+\\&\mu\sum_{k\geq1}(\chi_j\cdot\nabla_{\mathbf{Q}_j})\mathbf{g}_i^{(k)}\cos(k\Omega t)+\mu\sum_{k\geq1}(\rho_j\nabla_{\mathbf{P}_j})\mathbf{g}_i^{(k)}\cos(k\Omega t);\end{aligned} \tag{11}$$

$$\mu\,\dot{\rho}_i\ =-\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\partial H}{\partial\mathbf{Q}_i}-\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\partial H}{\partial\mathbf{Q}_i}+$$

$$\mu(\chi_j\cdot\nabla_{\mathbf{Q}_j})\frac{\mathbf{h}_i^{(0)}}{2}+\mu(\rho_j\cdot\nabla_{\mathbf{P}_j})\frac{\mathbf{h}_i^{(0)}}{2}+\sum_{k\geq 1}\mathbf{h}_i^{(k)}\cos(k\Omega t)+$$
$$\mu\sum_{k\geq 1}(\chi_j\cdot\nabla_{\mathbf{Q}_j})\mathbf{h}_i^{(k)}\cos(k\Omega t)+\mu\sum_{k\geq 1}(\rho_j\nabla_{\mathbf{P}_j})\mathbf{h}_i^{(k)}\cos(k\Omega t). \tag{12}$$

The terms in (11), (12) do not all have the same order. While the terms $\mu\,\dot{\chi}\sim\mu\Omega\chi\sim\omega_0\chi$, $\mu\,\dot{\rho}\sim\mu\Omega\rho\sim\omega_0\rho$ are not small, the terms with $\mu\chi$, $\mu\rho$ are much smaller. So, we find for the fast equations accurate up to zero-order terms in μ

$$\mu\,\dot{\chi_i}\simeq\sum_{k\geq 1}\mathbf{g}_i^{(k)}\cos(k\Omega t);\quad \mu\,\dot{\rho_i}\simeq\sum_{k\geq 1}\mathbf{h}_i^{(k)}\cos(k\Omega t). \tag{13}$$

Integrating these equations, we obtain

$$\chi_i\simeq\frac{1}{\mu\Omega}\sum_{k\geq 1}\frac{\mathbf{g}_i^{(k)}}{k}\sin(k\Omega t);\quad \rho_i\simeq\frac{1}{\mu\Omega}\sum_{k\geq 1}\frac{\mathbf{h}_i^{(k)}}{k}\sin(k\Omega t). \tag{14}$$

Subsequently, we solve the fast equations taking into account the terms of first-order in μ too, substituting (14) into (11), (12) and again integrating with respect to time (recall that $\mathbf{Q}_i$, $\mathbf{P}_i$, $H(\mathbf{Q},\mathbf{P},t)$ are held constant)

$$\chi_i\simeq\frac{1}{\mu\Omega}\sum_{k\geq 1}\frac{\mathbf{g}_i^{(k)}}{k}\sin(k\Omega t)-\frac{1}{\mu\Omega^2}\sum_{k\geq 1}\frac{1}{k^2}\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\cdot\left(\frac{\partial H}{\partial\mathbf{P}_i}+\frac{\mathbf{g}_i^{(o)}}{2}\right)\cdot\cos(k\Omega t)-$$
$$\frac{1}{\mu\Omega^2}\sum_{l,k\geq 1(l\neq k)}\frac{1}{2k(k-l)}\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\mathbf{g}_i^{(l)}\cos[(k-l)\Omega t]-$$
$$\frac{1}{\mu\Omega^2}\sum_{l,k\geq 1}\frac{1}{2k(k+l)}\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\mathbf{g}_i^{(l)}\cos[(k+l)\Omega t]; \tag{15}$$

$$\rho_i\simeq\frac{1}{\mu\Omega}\sum_{k\geq 1}\frac{\mathbf{h}_i^{(k)}}{k}\sin(k\Omega t)+\frac{1}{\mu\Omega^2}\sum_{k\geq 1}\frac{1}{k^2}\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\cdot\left(\frac{\partial H}{\partial\mathbf{Q}_i}-\frac{\mathbf{h}_i^{(0)}}{2}\right)\cdot\cos(k\Omega t)-$$
$$\frac{1}{\mu\Omega^2}\sum_{l,k\geq 1(l\neq k)}\frac{1}{2k(k-l)}\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\mathbf{h}_i^{(l)}\cos[(k-l)\Omega t]-$$
$$\frac{1}{\mu\Omega^2}\sum_{l,k\geq 1}\frac{1}{2k(k+l)}\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\mathbf{h}_i^{(l)}\cos[(k+l)\Omega t]. \tag{16}$$

Expressions (15), (16) represent a rapidly oscillating response of coordinates and momenta of the system to the fast external actions. The point is that it is not an unique reaction of the system. The mean trajectories $(\mathbf{Q},\mathbf{P})$ are altered too. Inserting (15), (16) into (9), (10) and averaging over the period $2\pi/\Omega$, we obtain the following equations of motion for the mean canonical coordinates:

$$\dot{\mathbf{Q}}_i=\left(\frac{\partial H}{\partial\mathbf{P}_i}+\frac{\mathbf{g}_i^{(0)}}{2}\right)-$$
$$\frac{1}{2\Omega^2}\sum_{k\geq 1}\frac{1}{k^2}\left\{\left(\mathbf{g}_{j'}^{(k)}\cdot\nabla_{\mathbf{Q}_{j'}}+\mathbf{h}_{j'}^{(k)}\cdot\nabla_{\mathbf{P}_{j'}}\right)\left[\left(\frac{\partial H}{\partial\mathbf{P}_j}+\frac{\mathbf{g}_j^{(0)}}{2}\right)\cdot\nabla_{\mathbf{Q}_j}-\left(\frac{\partial H}{\partial\mathbf{Q}_j}-\frac{\mathbf{h}_j^{(0)}}{2}\right)\cdot\nabla_{\mathbf{P}_j}\right]\right\}\mathbf{g}_i^{(k)}-$$
$$\frac{1}{2\Omega^2}\sum_{l\neq k\geq 1}\frac{1}{2k\,(l-k)}\left\{\left(\mathbf{g}_{j'}^{(l)}\cdot\nabla_{\mathbf{Q}_{j'}}+\mathbf{h}_{j'}^{(l)}\cdot\nabla_{\mathbf{P}_{j'}}\right)\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\right\}\mathbf{g}_i^{(|l-k|)}-$$
$$\frac{1}{2\Omega^2}\sum_{l,k\geq 1}\frac{1}{2k\,(l+k)}\left\{\left(\mathbf{g}_{j'}^{(l)}\cdot\nabla_{\mathbf{Q}_{j'}}+\mathbf{h}_{j'}^{(l)}\cdot\nabla_{\mathbf{P}_{j'}}\right)\left(\mathbf{g}_j^{(k)}\cdot\nabla_{\mathbf{Q}_j}+\mathbf{h}_j^{(k)}\cdot\nabla_{\mathbf{P}_j}\right)\right\}\mathbf{g}_i^{(l+k)}; \tag{17}$$

$$
\begin{aligned}
\dot{\mathbf{P}}_i \; = & -\left(\frac{\partial H}{\partial \mathbf{Q}_i}-\frac{\mathbf{h}_i^{(0)}}{2}\right)- \\
& \frac{1}{2\Omega^2}\sum_{k\geq 1}\frac{1}{k^2}\left\{\left(\mathbf{g}_{j'}^{(k)}\cdot\nabla_{\mathbf{Q}_{j'}}+\mathbf{h}_{j'}^{(k)}\cdot\nabla_{\mathbf{P}_{j'}}\right)\left[\left(\frac{\partial H}{\partial \mathbf{P}_j}+\frac{\mathbf{g}_j^{(0)}}{2}\right)\cdot\nabla_{\mathbf{Q}_j}-\left(\frac{\partial H}{\partial \mathbf{Q}_j}-\frac{\mathbf{h}_j^{(0)}}{2}\right)\cdot\nabla_{\mathbf{P}_j}\right]\right\}\mathbf{h}_i^{(k)}- \\
& \frac{1}{2\Omega^2}\sum_{l\neq k\geq 1}\frac{1}{2k\,(l-k)}\left\{\left(\mathbf{g}_{j'}^{(l)}\cdot\nabla_{\mathbf{Q}_{j'}}+\mathbf{h}_{j'}^{(l)}\cdot\nabla_{\mathbf{P}_{j'}}\right)\left(\mathbf{g}_{j}^{(k)}\cdot\nabla_{\mathbf{Q}_{j}}+\mathbf{h}_{j}^{(k)}\cdot\nabla_{\mathbf{P}_{j}}\right)\right\}\;\mathbf{h}_i^{(|l-k|)}- \\
& \frac{1}{2\Omega^2}\sum_{l,k\geq 1}\frac{1}{2k\,(l+k)}\left\{\left(\mathbf{g}_{j'}^{(l)}\cdot\nabla_{\mathbf{Q}_{j'}}+\mathbf{h}_{j'}^{(l)}\cdot\nabla_{\mathbf{P}_{j'}}\right)\left(\mathbf{g}_{j}^{(k)}\cdot\nabla_{\mathbf{Q}_{j}}+\mathbf{h}_{j}^{(k)}\cdot\nabla_{\mathbf{P}_{j}}\right)\right\}\mathbf{h}_i^{(l+k)}.
\end{aligned}
\tag{18}
$$

It is pertinent to note that all additional terms ($\sim 1/\Omega^2$) in these equations come from the last two terms of (9), (10) when integrating of the function $\cos(k'\Omega t)\cos(k\Omega t)$ over time: $\frac{\Omega}{2\pi}\int\limits_0^{2\pi/\Omega}\cos(k'\Omega t)\cos(k\Omega t)dt=\frac{1}{2}\delta_{kk'}$.

Relations (17), (18) constitute the effective equations of motion of a system subject to high-frequency external actions. The respective Liouville equation for the effective distribution function $F(\mathbf{Q},\mathbf{P},t)$ follows immediately from (17), (18) in the form (4)-(5):

$$
\frac{\partial\left(F\overline{\mathcal{J}}\right)}{\partial t}=-\nabla_{\mathcal{G}}\cdot\left(\dot{\mathcal{G}}\;F\overline{\mathcal{J}}\right);\quad \frac{d\overline{\mathcal{J}}}{dt}=-\overline{\mathcal{J}}\,\nabla_{\mathcal{G}}\cdot\dot{\mathcal{G}};
\tag{19}
$$

where $\mathcal{G}=(\mathbf{Q},\mathbf{P})$ is the phase space vector of the mean coordinates, $\overline{\mathcal{J}}$ is the inverse Jacobian of the transformation from the initial mean phase coordinates to the coordinates at time t:

$$
\overline{\mathcal{J}}=\frac{\partial\left(\mathbf{Q}_1^{(0)},...,\mathbf{Q}_N^{(0)},\mathbf{P}_1^{(0)},...,\mathbf{P}_N^{(0)}\right)}{\partial\left(\mathbf{Q}_1^{(t)},...,\mathbf{Q}_N^{(t)},\mathbf{P}_1^{(t)},...,\mathbf{P}_N^{(t)}\right)}.
\tag{20}
$$

The phase space velocity vector $\dot{\mathcal{G}}$ is given by the right-hand sides of the equations (17), (18).

In the simple case of a forced nonlinear oscillator with a Hamiltonian $H=p^2/2m+U(q)$ and the driving force $F(q)\cos\Omega t$, we have the only nonvanishing amplitude $h^{(1)}(q)=F(q)$ and from the equations (17), (18) follows the equation of motion in the mean coordinate

$$
m\,\ddot{Q}+\left(\frac{dU}{dq}\right)_Q+\frac{1}{2}\frac{F(Q)}{\Omega^2}\left(\frac{dF}{dq}\right)_Q=0.
\tag{21}
$$

This coincides with the result first obtained by Kapitza [1], [5] . The last term in (21) stabilizes the inverted pendulum. In other cases, there may be different extra terms in equations (17), (18) and respectively in the Liouville equation. These terms depend on various combinations of derivatives of the functions $\mathbf{g}_i^{(k)},\mathbf{h}_i^{(k)}$ with respect to canonical variables and may be responsible for many interesting dynamical and thermodynamic effects.

It is important to emphasize that even in the case when the equations (1), (2) are Hamiltonian, the equations (17), (18) may be non-Hamiltonian, i.e. the compressibility does not in general vanish $\varkappa(\mathcal{G})=\nabla_{\mathcal{G}}\,\dot{\mathcal{G}}\neq\mathbf{0}$. It is not surprising since a driven system behaves like a thermodynamically open system.

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